Fast Waveguide Mode Computation Using Wavelet-Like Basis Functions

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***Abstract*— The use of wavelet-like basis functions for solving electromagnetics problems is demonstrated. In particular, the modes of an arbitrarily shaped hollow metallic waveguide using a surface integral equation/method of moments (MOM) formulation are found. A class of wavelet-like basis functions is used to produce a sparse MOM impedance matrix, allowing the use of sparse matrix methods for fast solution of the problem. The same method applies directly to the external scattering problem. For the examples considered, the wavelet domain impedance matrix has about 20% nonzero elements, and the time required to compute its LU factorization is reduced by approximately a factor of 10 compared to the original full matrix.**

I. INTRODUCTION

RECENTLY, there has been a great deal of interest in using wavelets or wavelet-like basis functions to speed the solution of integral equations arising in electromagnetics [1]–[3]. Here, we show that use of the wavelet-like basis functions of Alpert *et al.* [4] accelerates the solution of surface integral equation/method of moments problems. This is illustrated by solving for the modes of an arbitrarily shaped hollow metallic waveguide. This problem results in a nonlinear eigenequation, the solution of which requires many evaluations of the determinant of an MOM impedance matrix [5]. If pulse basis functions are used to expand the current on the waveguide wall, the resulting MOM matrix $\boldsymbol{Z}$ will be full. This is because pulse basis currents radiate well, and hence they all interact with one another. However, the impedance matrix is sparse when represented in the wavelet-like basis of [4]. This is because many of the wavelet-like basis functions are poor radiators and hence do not interact with one another. The determinant of the new sparse impedance matrix $\boldsymbol{Z}'$ may be computed rapidly using sparse matrix methods. This technique may also be applied directly to solving the exterior scattering problem, since the same impedance matrix is used.

II. THEORY

The TM modes of a hollow cylindrical waveguide with boundary contour C are characterized by the field component E_z satisfying the boundary condition $E_z = 0$ on C. Enforcing this boundary condition yields the integral eigenequation

$$\oint_C dl\ g_0(\boldsymbol{r}_s, \boldsymbol{r}'_s; k_s) J_z(\boldsymbol{r}_s) = 0, \qquad \boldsymbol{r}'_s \in C. \tag{1}$$

In (1), J_z is the current on the waveguide wall, g_0 is the two-dimensional Green's function $[g_0(\boldsymbol{r}_s, \boldsymbol{r}'_s; k_s) = (i/4)H_0^{(1)}(k_s|\boldsymbol{r}_s - \boldsymbol{r}'_s|)]$, and the eigenvalues k_s are the cutoff wavenumbers of the TM modes. The s subscript denotes transverse-to-z coordinates.

Expanding the surface current $J_z(\boldsymbol{r}_s), \boldsymbol{r}_s \in C$, in rectangular pulse basis functions and point matching the resulting equation at the centers of the pulses gives the MOM matrix formulation $\boldsymbol{Z}(k_s) \cdot \boldsymbol{a} = 0$, where $\boldsymbol{a}$ is a vector containing expansion coefficients for the surface current and $\boldsymbol{Z}(k_s)$ is an impedance matrix with elements which may be approximated numerically by a symmetric matrix

$$Z_{ij}(k_s) = \begin{cases} \Delta_i^2\left[1 + i\frac{2}{\pi}\ln\left(\frac{e^{\gamma}k_s\Delta_i}{4e}\right)\right], & i = j \\ \Delta_i\Delta_j H_0^{(1)}(k_s|\boldsymbol{r}_i - \boldsymbol{r}_j|), & i \neq j, \end{cases} \tag{2}$$

where Δ_i and $\boldsymbol{r}_i$ are the width and center position of the ith pulse, and $\gamma = 0.5772$ is Euler's constant. The eigenvalues k_s are then determined from the nonlinear eigenequation $\det[\boldsymbol{Z}(k_s)] = 0$. This equation may be solved using a nonlinear root finding algorithm, requiring many evaluations of the left hand side. Examples are shown in Fig. 1. Since $\boldsymbol{Z}$ is generally a full matrix, computing its determinant by LU factorization is expensive, with a computational cost of $O(N^3)$, where N is the dimension of $\boldsymbol{Z}$.

To expedite the computation, we make use of the wavelet-like basis functions introduced by Alpert *et al.* [4]. These bases are designed specifically to produce sparse representations

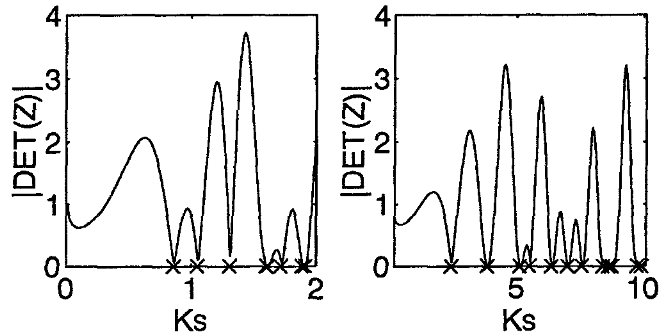

Fig. 1. Determinant of MOM impedance matrix $\boldsymbol{Z}$ for (a) a rectangular waveguide, sides 9m by 4m, modeled with $N = 52$ pulse basis functions and (b) a circular waveguide, radius 1 m, modeled with $N = 50$ pulses. True eigenvalues are marked with a ×.

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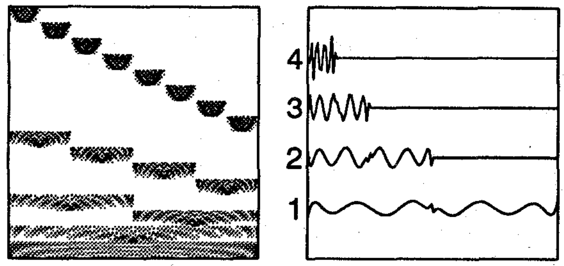

Fig. 2. (a) Magnitude of basis vector matrix $\boldsymbol{U}$, with $N = 128$ discretization points and $k = 8$ vanishing moments. Each row is one basis vector. (b) Sample basis vectors: rows 4, 68, 100, and 116 of $\boldsymbol{U}$.

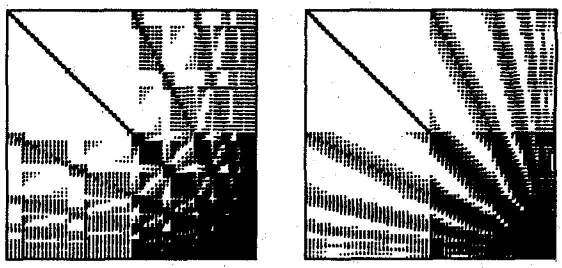

Fig. 3 Sparsity structure of new MOM matrix $\boldsymbol{Z}'$ for (a) a rectangular waveguide, aspect ratio $= 9/4$, and (b) a circular waveguide. Both are originally modeled with $N = 512$ pulse basis functions of width $\Delta \approx \lambda/10$. The matrices are (a) 22.6% and (b) 18.8% full.

of integral operators having smooth, nonoscillatory (Laplace-like) kernels. However, their use also produces significant sparseness in the representation of the integral operator of (1), with a Hankel function kernel.

These orthonormal basis vectors have two principal properties. First, they have support on different length scales in space. Second, all but k basis vectors have k vanishing moments, where k is an integer chosen by the user. That is, most of the basis vectors are orthogonal to polynomials of degree $< k$, and hence have high spatial frequency content. Fig. 2 shows the basis vectors for $N = 128$ discretization points and $k = 8$ vanishing moments. A detailed description of their construction is given in [4].

Because many of the new basis vectors have high spatial frequency content, they are poor radiators/receivers and hence do not interact with one another. This accounts for the sparseness of the new impedance matrix $\boldsymbol{Z}'$. If $\boldsymbol{U}$ is a matrix with the new basis vectors as its rows, then $\boldsymbol{Z}$ is represented in the new basis by the similarity transformation $\boldsymbol{Z}' = \boldsymbol{U}\boldsymbol{Z}\boldsymbol{U}^T$. Because of the special structure of $\boldsymbol{U}$, the similarity transformation above can be shown to have a computational cost of $O(N^2)$. The new matrix $\boldsymbol{Z}'$ is thresholded by zeroing elements Z'_{ij} with magnitude less than a cutoff τ. Following [4], τ is chosen to be $\tau = \epsilon \|\boldsymbol{Z}\|_\infty / N$, where ϵ is a constant, N is the dimension of $\boldsymbol{Z}$, and $\|\boldsymbol{Z}\|_\infty$ is the infinity or row-sum norm of $\boldsymbol{Z}$, $\|\boldsymbol{Z}\|_\infty = \max_i \sum_{j=1}^{N} |Z_{ij}|$.

III. Results

In the cases we have examined, the transformed and thresholded operator $\boldsymbol{Z}'$ is a sparse matrix with αN^2 nonzero elements, where typically $\alpha \approx 0.2$. Fig. 3 shows the sparsity structure of $\boldsymbol{Z}'$ for a rectangular and circular waveguide. In this example and in the following results we have used the parameters $k = 8$ vanishing moments and $\epsilon = 10^{-2}$ for thresholding. These seem to provide near optimal sparsity without sacrificing the accuracy of the solution (the maximum error in the calculated determinant is $\approx 0.5\%$).

Using a sparse matrix package, $\det[\boldsymbol{Z}'(k_s)]$ can be computed by LU factorization in much less time than is required for the full, pulse basis matrix $\boldsymbol{Z}$. Fig. 4 summarizes the computer time required for the case of a circular cylinder modeled by N basis functions. Similar results have been observed for a rectangular waveguide with aspect ratio 9/4, and for both the circular and rectangular waveguides with

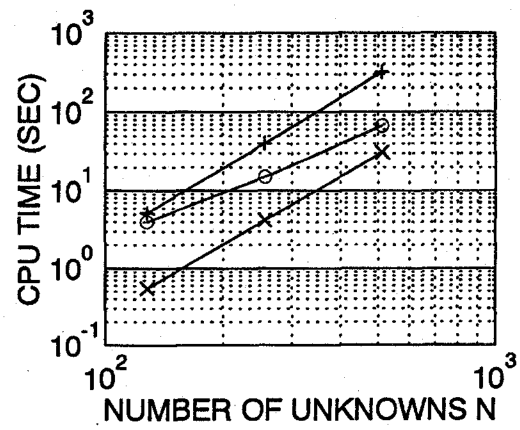

Fig. 4. Timing data for the circular waveguide, $\lambda/10$ discretization: factor full matrix $\boldsymbol{Z}$ (+), similarity transform $\boldsymbol{Z}' = \boldsymbol{U}\boldsymbol{Z}\boldsymbol{U}^T$ (o), factor sparse matrix $\boldsymbol{Z}'$ (×).

TE (H_z) polarization. In each case the discretization is such that $\Delta_i \approx 0.1\lambda$. The CPU time is given for a SPARC 2 workstation, using double precision. The sparse matrix package used is Sparse1.3 [6]. Notice the factor of 10 speed increase in the factorization step. The additional overhead of the similarity transformation $\boldsymbol{Z}' = \boldsymbol{U}\boldsymbol{Z}\boldsymbol{U}^T$ has a lower computational complexity than the factorization step, and so its cost will become insignificant as N becomes very large.

IV. Conclusion

We have demonstrated the use of the wavelet-like basis functions of [4] to reduce the MOM impedance matrix $\boldsymbol{Z}$ for a hollow metallic waveguide from N^2 nonzero elements in the pulse basis to αN^2 nonzero elements in the wavelet-like basis, where $\alpha \approx 0.2$ for the examples considered. This result also applies directly to the external scattering problem. The new sparse matrix can be factored and its determinant computed much more rapidly than the original full matrix. For the example above, the speedup factor is about 10. Possibilities for the future include using wavelet-like bases to rapidly solve for scattering from three-dimensional surfaces or from volume scatterers. In addition, as indicated in [4], it may be possible to construct modified basis functions that will produce even more sparsity in the impedance matrix, further increasing the solution efficiency.

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